OPTIMAL GAMBLING SYSTEMS

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Until recently, the mathematics of optimal gambling has been primarily concerned with the formulation and proof of variants of the following fact. The expected terminal fortune of a gambler, whose fortune is always nonnegative, engaged in any succession of gambles that are fair or unfair to him cannot be larger than his initial fortune.

These results, though important, do not say how well the gambler can do in given unfair gambling situations, nor do they suggest what gambling strategy is best in his bad situation. These latter problems have their own mathematical interest, and the underlying ideas used in attacking them can be applied to establish various inequalities for wide classes of stochastic processes of theoretical, and perhaps practical, interest.

Some illustrations of our results in this area, to be published elsewhere, are here stated without proof in language that, for brevity, is somewhat informal.

Let r and w be two numbers less than 1. Suppose a gambler whose initial fortune is positive can make a sequence of bets, on any one of which he can stake any amount then in his possession. Each bet results in either a gain or a loss, where the ratio of the possible gain to the possible loss is as $1 - r$ to r, the gain occurs with probability w, and the loss occurs with probability $1 - w$.

Theorem 1. *If $w > r$, there exists k, $0 < k < 1$, such that a gambler who stakes cy whenever his fortune is y will have his fortune converge almost certainly to $+\infty$ or 0, according as $0 < c < k$ or $c > k$. If $c = k$, the* lim sup *and* lim inf *of his fortunes will almost certainly be $+\infty$ and 0, respectively.*

Let a certain positive G be a gambler's goal. Roughly speaking, a betting strategy is *optimal* if it maximizes the probability that the gambler's fortune ever attains the goal. A gambler *plays boldly* if he always stakes as much as possible consistent with the condition that his fortune never exceed G or become negative.

Theorem 2. *If $w < r$, bold play is optimal. Bold play is not the only optimal strategy.*

Let g be a random variable bounded below by -1 and of expectation 0—a fair lottery, so to speak—and let g_n, $n = 1, 2, \ldots$, be independent, each with the same distribution as g. For any x, $0 < x < 1$, consider any stochastic process f_n, $n = 0, 1, 2, \ldots$—the fortune of a gambler at time n—such that: (i) $f_0 = x$; (ii) $f_n \geq 0$ for all n; (iii) $f_{n+1} = f_n + s_{n+1}g_{n+1}$, where s_n—the stake at time n—is nonnegative and may depend upon $f_0, \ldots, f_n$. Then $\sigma = s_1, s_2, \ldots$ is a *strategy* and $p(\sigma, x)$ is the probability that for some n, $f_n \geq 1$. Before stating a main result, we note that well-known theorems easily imply: (iv) For all σ, $p(\sigma, x) \leq x$; (v) if g is bounded above and if its upper bound is attained with positive probability, then there is a σ such that $p(\sigma, x) = x$; (vi) if g is bounded above but its upper bound is not attained with positive probability, then for all σ and x, $p(\sigma, x) < x$ but the sup of $p(\sigma, x)$ over all σ is x.

Suppose now that g is unbounded above. It is relatively easy to exhibit a g

such that $U(x)$, the supremum of $p(\sigma,x)$ over all σ, is x. It is much more difficult to see, as Donald Ornstein was the first to do, that there exist g for which $U(x) < x$. We now know more:

THEOREM 3. *Suppose that g has mean* 0 *and that* $|g|$ *has positive mean. Then in order that* $U(x) = x$ *it is necessary and sufficient that the following condition hold: The limit inferior, as z approaches* ∞, *of* $(-z \int_{g<z} g)(\int_{g<z} g^2)^{-1}$ *is* 0.

This research owes much to many, both for ideas and for financial support, as will be explained in the full publication.

FIELDS OF UNIT VECTORS IN THE FOUR-SPACE OF GENERAL RELATIVITY

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1. A previous paper[1] deals with the tangent vectors to the minimal geodesics of the four-space of general relativity V_4, which, according to Einstein, are the paths of light.[2] This means that the tensor g_{ij} in the fundamental equation

$$ds^2 = g_{ij}dx^i dx^j \tag{1}$$

is not positive definite. The right-hand member of this equation stands for the sum of terms as i and j take the values 1 to 4.

This convention is used throughout this paper, namely, that when within a term the same letter enters as a superscript and subscript, the term stands for the sum of terms as the index takes the values 1 to 4.

The present paper deals with unit vectors in the V_4, that is,

$$\lambda_{1;}{}^{i}\lambda_{1;\,i} = 1, \quad \lambda_{2;}{}^{i}\lambda_{2;\,i} = 1, \quad \lambda_{3;}{}^{i}\lambda_{3;\,i} = 1, \tag{2}$$

which are mutually orthogonal, that is,[3]

$$\lambda_{1;}{}^{i}\lambda_{2;\,i} = 0, \quad \lambda_{2;}{}^{i}\lambda_{3;\,i} = 0, \quad \lambda_{3;}{}^{i}\lambda_{1;\,i} = 0. \tag{3}$$

2. We use the null vector λ_i of the previous paper, that is,

$$\lambda_i\lambda^i = 0, \tag{4}$$

and require that the vectors $\lambda_{1;\,i}$, $\lambda_{2;\,i}$, $\lambda_{3;\,i}$ be orthogonal to the vector λ_i, that is,

$$\lambda_{1;\,i}\lambda^i = \lambda_{1;}{}^{i}\lambda_i = 0, \ \lambda_{2;\,i}\lambda^i = \lambda_{2;}{}^{i}\lambda_i = 0, \quad \lambda_{3;\,i}\lambda^i = \lambda_{3;}{}^{i}\lambda_i = 0. \tag{5}$$

We put

$$\lambda_{1;\,i,j} = \lambda_i\lambda_{i;\,j}, \quad \lambda_{2;\,i,j} = \lambda_1\lambda_{2;\,j}, \quad \lambda_{3;\,i,j} = \lambda_i\lambda_{3;\,j}, \tag{6}$$

where the terms on the left are the covariant derivatives of $\lambda_{1;\,i}$, $\lambda_{2;\,i}$, $\lambda_{3;\,i}$ with respect to x^j.[4]

This convention is used throughout this paper, namely, a component of a vector